A family of measures associated with iterated function systems

Palle E. T. Jorgensen*
Department of Mathematics
University of Iowa
Iowa City, Iowa 52242

February 8, 2008

Abstract

Let (X, d) be a compact metric space, and let an iterated function system (IFS) be given on X, i.e., a finite set of continuous maps $\sigma_i \colon X \to X$, $i = 0, 1, \cdots, N-1$. The maps σ_i transform the measures μ on X into new measures μ_i. If the diameter of $\sigma_{i_1} \circ \cdots \circ \sigma_{i_k}(X)$ tends to zero as $k \to \infty$, and if $p_i > 0$ satisfies $\sum_i p_i = 1$, then it is known that there is a unique Borel probability measure μ on X such that

$$\mu = \sum_i p_i \ \mu_i \tag{*}$$

In this paper, we consider the case when the p_is are replaced with a certain system of sequilinear functionals. This allows us to study the variable coefficient case of (*), and moreover to understand the analog of (*) which is needed in the theory of wavelets.

1 Introduction

A finite system of continuous functions $\sigma_i \colon X \to X$ in a compact metric space X is said to be an *iterated function system* (IFS) if there is a mapping $\sigma \colon X \to X$, onto X, such that

$$\sigma \circ \sigma_i = id_X \tag{1.1}$$

If there is a constant $0 < c < 1$ such that

$$d(\sigma_i(x), \sigma_i(y)) \leq c\ d(x, y),\ \ x, y \in X, \tag{1.2}$$

then we say that the IFS is *contractive*. In that case, there is, for every configuration $p_i > 0$, $\sum_i p_i = 1$, a unique Borel probability measure μ, $\mu = \mu_{(p)}$ on X such that

$$\mu = \sum p_i \ \mu \circ \sigma_i^{-1}. \tag{1.3}$$

*Work supported in part by the NSF.

This follows from a theorem of Hutchinson [4]. The mappings σ_i might be defined initially on some Euclidean space E. If the contractivity (1.2) is assumed, then there is a unique compact subset $X \subset E$ such that

$$X = \bigcup_i \sigma_i(X), \tag{1.4}$$

and this set X is the support of μ.

Example 1.1 *Let $E = \mathbb{R}$, $\sigma_0(x) = \frac{x}{3}$, $\sigma_1 = \frac{x+2}{3}$, $p_0 = p_1 = \frac{1}{2}$. In this case, X is the familiar middle-third Cantor set, and μ is the Cantor measure supported in X with Hausdorff dimension $d = \frac{\ln 2}{\ln 3}$. But at the same time X may be identified with the compact Cartesian product $X \cong \prod \mathbb{Z}_2 = \mathbb{Z}_2^{\mathbb{N}}$, where $\mathbb{Z}_2 = \mathbb{Z}\diagup 2\mathbb{Z} = \{0,1\}$, and $\mathbb{N} = \{0,1,2,\cdots\}$, and μ is the infinite product measure on $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots$ with weights $(\frac{1}{2}, \frac{1}{2})$ on each factor.*

Example 1.2 *Let $E = \mathbb{R}$, $\sigma_0(x) = \frac{x}{2}$, $\sigma_1(x) = \frac{x+1}{2}$, $p_0 = p_1 = \frac{1}{2}$. In this case, $X = [0,1]$, i.e., the compact unit interval, and μ is the restriction to $[0,1]$ of the usual Lebesgue measure dt on $\mathbb{R}$.*

Let $\mathbb{T} = \mathbb{R}\diagup 2\pi\mathbb{Z} = \{z \in \mathbb{C} \mid |z| = 1\}$ be the usual torus. Let $N \in \mathbb{N}$, $N \geq 2$, and let m_i: $\mathbb{T} \to \mathbb{C}$, $i = 0, 1, \cdots, N-1$ be a system of L^∞-functions such that the $N \times N$ matrix

$$\frac{1}{\sqrt{N}} \left(m_j \left(z e^{i 2\pi \frac{k}{N}} \right) \right)_{j,k=0}^{N-1} \qquad , z \in \mathbb{T} \tag{1.5}$$

is unitary. Set

$$S_j f(z) = m_j(z) f(z^N) \qquad , z \in \mathbb{T},\ f \in L^2(\mathbb{T}). \tag{1.6}$$

Then it is well known [7] that the operators S_j satisfy the following two relations,

$$S_j^* S_k = \delta_{j,k} I \tag{1.7}$$

$$\sum_j S_j S_j^* = I \tag{1.8}$$

where I denotes the identity operator in the Hilbert space $\mathcal{H}:= L^2(\mathbb{T})$. The converse implication also holds, see [2]. Systems of isometries satisfying (1.7) – (1.8) are called *representations of the Cuntz algebra $\mathcal{O}_N$*, see [3], and the particular representations (1.6) are well known to correspond to multiresolution wavelets; the functions m_j are denoted wavelet filters. These same functions are used in subband filters in signal processing, see [2].

It is easy to see that there is a unique Borel measure P on $[0,1)$ taking values in the orthogonal projections of $\mathcal{H}$ such that

$$P\left(\left[\frac{i_1}{N} + \cdots + \frac{i_k}{N^k}, \frac{i_1}{N} + \cdots + \frac{i_k}{N^k} + \frac{1}{N^k} \right) \right) = S_{i_1} \cdots S_{i_k} S_{i_k}^* \cdots S_{i_1}^*. \tag{1.9}$$

Let

$$e_n(z) = z^n,\ z \in \mathbb{T}, n \in \mathbb{Z}. \tag{1.10}$$

Example 1.3 *Let $N \in \mathbb{N}$, $N \geq 2$, and set $m_j(z) = z^j$, $0 \leq j < N$. Then (1.5) is satisfied, and we have*

$$\begin{cases} S_0^* e_0 = e_0 \\ S_j^* e_0 = 0 \qquad , \ 0 < j < N. \end{cases}$$

It follows easily that the corresponding measure

$$\mu_0(\cdot) := \|P(\cdot)e_0\|^2 \tag{1.11}$$

on $[0,1)$ is the Dirac measure δ_0 at $x = 0$, i.e.,

$$\delta_0(E) = \begin{cases} 1 & \text{if } 0 \in E \\ 0 & \text{if } 0 \notin E \end{cases}. \tag{1.12}$$

Here $P(\cdot)$ refers to the projection valued measure determined by (1.9) when the representation of $\mathcal{O}_N$ is specified by the system $m_j := e_j$, $0 \leq j < N$.

Example 1.4 *Let $N \in \mathbb{N}$, $N \geq 2$, and set*

$$m_j(z) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{i2\pi \frac{jk}{N}} \ z^k. \tag{1.13}$$

Again the condition (1.5) is satisfied, and one checks that

$$S_j^* e_0 = \frac{1}{\sqrt{N}} e_0 \qquad , 0 \leq j < N; \tag{1.14}$$

and now the measure $\mu_0(\cdot) = \|P(\cdot)e_0\|^2$ on $[0,1)$ is the restriction to $[0,1)$ of the Lebesgue measure on $\mathbb{R}$. It is well known that the wavelet corresponding to (1.13) is the familiar Haar wavelet corresponding to N-adic subdivision, see [2]. It is also known that generally, for wavelets other than the Haar systems, the corresponding representations (1.6) of $\mathcal{O}_N$ does not admit a simultaneous eigenvector f, i.e., there is no solution $f \in \mathcal{H} \setminus \{0\}$, $\lambda_j \in \mathbb{C}$ to the joint eigenvalue problem

$$S_j^* f = \lambda_j f \qquad , 0 \leq j < N. \tag{1.15}$$

Proposition 1.5 *Let $N \in \mathbb{N}$, $N \geq 2$, and let $(S_j)_{0 \leq j < N}$ be a representation of $\mathcal{O}_N$ on a Hilbert space $\mathcal{H}$. Suppose there is a solution $f \in \mathcal{H}$, $\|f\| = 1$, to the eigenvalue problem (1.15) for some $\lambda_j \in \mathbb{C}$. Then $\sum |\lambda_j|^2 = 1$, and the measure $\mu := \|P(\cdot)f\|^2$ satisfies*

$$\mu = \sum_{j=0}^{N-1} |\lambda_j|^2 \, \mu \circ \sigma_j^{-1} \tag{1.16}$$

where $\sigma_j(x) = \frac{x+j}{N}$, $\mu \circ \sigma_j^{-1}(E) = \mu(\sigma_j^{-1}(E))$ for Borel sets $E \subset [0,1)$, and $\sigma_j^{-1}(E) := \{x \mid \sigma_j(x) \in E\}$.

Proof. The reader may prove the proposition directly from the definitions, but the conclusion may also be obtained as a special case of the theorem in the next section. ∎

2 Measures And Iterated Function Systems

Let (X, d) be a compact metric space, and let $(\sigma_j)_{0 \le j < N}$ be an N-adic iterated function system (IFS). We say that the system is *complete* if

$$\lim_{k \to \infty} \text{diameter } (\sigma_{i_1} \circ \cdots \circ \sigma_{i_k}(X)) = 0, \tag{2.1}$$

We say that the IFS is *non-overlapping* if for each k the sets

$$A_k(i_1, \cdots, i_k) \colon = \sigma_{i_1} \circ \cdots \circ \sigma_{i_k}(X) \tag{2.2}$$

are disjoint, i.e., for every k, the sets $A_k(i_1, \cdots, i_k)$ are mutually disjoint for different multi-indices, i.e., different points in

$$\underbrace{\mathbb{Z}_N \times \cdots \times \mathbb{Z}_N}_{k\text{-times}}$$

where $\mathbb{Z}_N \colon = \{0, 1, \cdots, N-1\}$.

Remark 2.1 *It is immediate that, if a given IFS $(\sigma_j)_{0 \le j < N}$ arises as a system of distinct branches of the inverse of a single mapping $\sigma \colon X \to X$, i.e., if $\sigma(\sigma_i(x)) = x$ for $x \in X$, and $0 \le i < N$, then the partition system $\sigma_{i_1} \circ \cdots \circ \sigma_{i_k}(X)$ is non-overlapping.*

Theorem 2.2 *Let $N \in \mathbb{N}$, $N \ge 2$, and a Hilbert space $\mathcal{H}$. Let $(\sigma_j)_{0 \le j < N}$ be an IFS which is complete and non-overlapping. Then there is a unique projection valued measure P defined on the Borel subsets of X such that*

$$P\left(A_k(i_1, \cdots, i_k)\right) = S_{i_1} \cdots S_{i_k} S_{i_k}^* \cdots S_{i_1}^*. \tag{2.3}$$

This measure satisfies:
(a) $P(E) = P(E)^* = P(E)^2$, $E \in \mathcal{B}(X) =$ *the Borel subsets of* X.
(b) $\int_X dP(x) = I_{\mathcal{H}}$
(c) $P(E)P(F) = 0$ *if* $E, F \in \mathcal{B}(X)$ *and* $E \cap F = \varnothing$.
(d) $\sum_{j=0}^{N-1} S_j P(\sigma_j^{-1}(E)) S_j^* = P(E)$, $E \in \mathcal{B}(X)$.

It follows in particular that, for every $f \in \mathcal{H}$, the measure $\mu_f(\cdot) \coloneqq \|P(\cdot)f\|^2$ satisfies

$$\sum_{j=0}^{N-1} \mu_{S_j^* f} \circ \sigma_j^{-1} = \mu_f, \tag{2.4}$$

or equivalently

$$\sum_{j=0}^{N-1} \int_X \psi \circ \sigma_j \, d\mu_{S_j^* f} = \int_X \psi \, d\mu_f \tag{2.5}$$

for all bounded Borel functions ψ on X.

Corollary 2.3 *Let $N \in \mathbb{N}$, $N \geq 2$, be given, and consider a representation $(S_j)_{0 \leq j < N}$ of $\mathcal{O}_N$, and an associated IFS which is complete and non-overlapping. Let $P(\cdot)$ be the corresponding projection valued measure, i.e.,*

$$P\left(\sigma_{i_1} \circ \cdots \circ \sigma_{i_k}(X)\right) = S_{i_1} \cdots S_{i_k} S_{i_k}^* \cdots S_{i_1}^*. \tag{2.6}$$

For $f \in \mathcal{H}$, $\|f\| = 1$, set $\mu_f(\cdot) := \|P(\cdot)f\|^2$. Let $\mathfrak{A}$ be the abelian C^*-algebra generated by the projections $S_{i_1} \cdots S_{i_k} S_{i_k}^* \cdots S_{i_1}^*$ and let $\mathcal{H}_f$ be the closure of $\mathfrak{A}f$. Then there is a unique isometry $V_f \colon L^2(\mu_f) \to \mathcal{H}_f$ of $L^2(\mu_f)$ onto $\mathcal{H}_f$ such that

$$V_f(1) = f, \tag{2.7}$$

and

$$V_f M_{\chi_{A_k(i)}} V_f^* = S_{i_1} \cdots S_{i_k} S_{i_k}^* \cdots S_{i_1}^* \tag{2.8}$$

where $M_{\chi_{A_k(i)}}$ is the operator which multiplies by the indicator function of

$$A_k(i) := \sigma_{i_1} \circ \cdots \circ \sigma_{i_k}(X). \tag{2.9}$$

Proof. (Theorem 2.2) We refer to the paper [6] for a more complete discussion. With the assumptions, we note that for every k, and every multi-index $i = (i_1, \cdots, i_k)$ we have an abelian algebra of functions $\mathcal{F}_k$ spanned by the indicator functions $\chi_{A_k(i)}$ where $A_k(i) := \sigma_{i_1} \circ \cdots \circ \sigma_{i_k}(X)$. Since, for every k, we have the non-overlapping unions

$$\bigcup_i A_{k+1}(i_1, i_2, \cdots, i_k, i) = A_k(i_1, \cdots, i_k), \tag{2.10}$$

there is a natural embedding $\mathcal{F}_k \subset \mathcal{F}_{k+1}$. We wish to define the projection valued measure P as an operator valued map on functions on X in such a way that $\int_X \chi_{A_k(i)} dP = S_{i_1} \cdots S_{i_k} S_{i_k}^* \cdots S_{i_1}^*$. This is possible since the projections $S_{i_1} \cdots S_{i_k} S_{i_k}^* \cdots S_{i_1}^*$ are mutually orthogonal when k is fixed, and $(i_1, \cdots, i_k)$ varies over $(\mathbb{Z}_N)^k$. In view of the inclusions

$$\mathcal{F}_k \subset \mathcal{F}_{k+1}, \tag{2.11}$$

it follows that $\bigcup_k \mathcal{F}_k$ is an algebra of functions on X. Since the N-adic subdivision system $\{A_k(i) \mid k \in \mathbb{N},\ i \in (\mathbb{Z}_N)^k\}$ is complete, it follows that every continuous function on X is the uniform limit of a sequence of functions in $\bigcup_k \mathcal{F}_k$. Using now a standard extension procedure for measures, we conclude that the projection valued measure $P(\cdot)$ exists, and that it has the properties listed in the theorem. The reader is referred to [8] for additional details on the extension from $\bigcup_k \mathcal{F}_k$ to the Borel function on X. ■

Proof of Corollary 2.3. Let the systems (S_j) and (σ_j) be as in the statement of the theorem. To define $V_f \colon L^2(\mu_f) \to \mathcal{H}_f$, we set

$$V_f(1) = f, \text{and } V_f \chi_{A_k(i)} = S_{i_1} \cdots S_{i_k} S_{i_k}^* \cdots S_{i_1}^* f. \tag{2.12}$$

It is clear from the theorem that V_f defined this way extends to an isometry of $L^2(\mu_f)$ onto $\mathcal{H}_f$, and a direct verification reveals that the covariance relation (2.8) is satisfied.

It remains to prove (d) in theorem 2.2, or equivalently to prove (2.4) for every $f \in \mathcal{H}$. The argument is based on the same approximation procedure as we used above, starting with the algebra $\bigcup_k \mathcal{F}_k$. Note that

$$\begin{aligned} & \int_X \chi_{A_k(\alpha_1,\cdots,\alpha_k)}(\sigma_i(x))\, d\mu_{S_i^* f}(x) \\ = \; & \delta_{i,\alpha_1} \int_X \chi_{A_k(\alpha_1,\cdots,\alpha_k)}(x)\, d\mu_{S_i^* f}(x) \\ = \; & \delta_{i,\alpha_1} \left\| S_{\alpha_k}^* \cdots S_{\alpha_2}^* S_i^* f \right\|^2 \\ = \; & \delta_{i,\alpha_1} \int_X \chi_{A_k(\alpha)}(x)\, d\mu_f(x). \end{aligned}$$

Summing over i, we get

$$\sum_i \int_X \chi_{A_k(\alpha)} \circ \sigma_i \, d\mu_{S_i^* f} = \int_X \chi_{A_k(\alpha)} d\mu_f = \left\| S_{\alpha_k}^* \cdots S_{\alpha_1 i}^* f \right\|^2 .$$

The desired identity (2.5) now follows by yet another application of the standard approximation argument which was used in the proof of the first part of the theorem. ■

The simplest subdivision system is the one where the subdivisions are given by the N-adic fractions $\frac{\alpha_1}{N} + \frac{\alpha_2}{N^2} + \cdots + \frac{\alpha_k}{N^k}$ where $\alpha_i \in \mathbb{Z}_N = \{0, 1, \cdots, N-1\}$. Setting $\sigma_j(x) = \frac{x+j}{N}$, $j \in \mathbb{Z}_N$, we note that

$$\sigma_{\alpha_1} \circ \cdots \circ \sigma_{\alpha_k}([0,1)) = \left[\frac{\alpha_1}{N} + \cdots + \frac{\alpha_k}{N^k}, \frac{\alpha_1}{N} + \cdots + \frac{\alpha_k}{N^k} + \frac{1}{N^k} \right) .$$

As a result both the projection valued measure $P(\cdot)$ and the individual measures $\mu_f(\cdot) = \|P(\cdot) f\|^2$ are defined on the Borel subsets of $[0,1)$. If $\hat{P}(t) := \int_0^1 e^{itx} dP(x)$, then $\left\langle f \mid \hat{P}(t) f \right\rangle = \hat{\mu}_f(t)$ is the usual Fourier transform of the measure μ_f for $f \in \mathcal{H}$.

Moreover, by the Spectral theorem [8], there is a selfadjoint operator D with spectrum contained in $[0,1]$ such that

$$\widehat{P}(t) = e^{itD},$$

see [8]. In fact, the spectrum of D is equal to the support of the projection valued measure $P(\cdot)$.

Corollary 2.4 *Suppose the N-adic partition system used in the theorem is given by the N-adic fractions as in 2.10. Then the Fourier transform*

$$\hat{\mu}_f(t) = \int_0^1 e^{itx}\, d\mu_f(x) \qquad , t \in \mathbb{R} \tag{2.13}$$

of the measure $\mu_f(\cdot) = \|P(\cdot)f\|^2$ satisfies

$$\hat{\mu}_f(t) = \sum_{k=0}^{N-1} e^{i\frac{kt}{N}} \; \widehat{\mu_{S_k^* f}}(t/N). \tag{2.14}$$

Proof. With the N-adic subdivisions of the unit interval, the maps σ_k are $\sigma_k(x) = \frac{x+k}{N}$ for $k \in \mathbb{Z}_N = \{0, 1, \cdots, N-1\}$. Setting $\psi_t(x) = e^{itx}$ in (2.4), the desired result (2.14) follows immediately. ■

In the next result we show that for every $k \in \mathbb{N}$, there is an approximation formula for the Fourier transform $\hat{\mu}_f$ of the measure $\mu_f(\cdot) = \|P(\cdot)f\|^2$ involving the numbers $\left\|S_{\alpha_k}^* \cdots S_{\alpha_1}^* f\right\|^2$ as the multi-index $\alpha = (\alpha_1, \cdots, \alpha_k)$ ranges over $(\mathbb{Z}_N)^k$.

Corollary 2.5 *Let $N \in \mathbb{N}$, $N \geq 2$ be given. Let $(S_j)_{0 \leq j < N}$ be a representation of $\mathcal{O}_N$ on a Hilbert space $\mathcal{H}$, and let $P(\cdot)$ be the corresponding projection-valued measure defined on $\mathcal{B}([0,1))$. Let $f \in \mathcal{H}$, $\|f\| = 1$, and set $\mu_f(\cdot) = \|P(\cdot)f\|^2$. Then, for every k, we have the approximation*

$$\left| \hat{\mu}_f(t) - \sum_{\alpha_1, \cdots, \alpha_k} e^{it\left(\frac{\alpha_1}{N} + \cdots + \frac{\alpha_k}{N^k}\right)} \|S_\alpha^* f\|^2 \right| \leq |t| \, N^{-k} \tag{2.15}$$

where the summation is over multi-indices from $(\mathbb{Z}_N)^k$, and $S_\alpha^* := S_{\alpha_k}^* \cdots S_{\alpha_1}^*$.

Proof. A k-fold iteration of formula (2.14) from the previous corollary yields,

$$\hat{\mu}_f(t) = \sum_{\alpha_1, \cdots, \alpha_k} e^{it\left(\frac{\alpha_1}{N} + \cdots + \frac{\alpha_k}{N^k}\right)} \widehat{\mu_{S_\alpha^* f}}(t/N^k)$$

and

$$\widehat{\mu_{S_\alpha^* f}}(t/N^k) - \|S_\alpha^* f\|^2 = \int_0^1 \left(e^{itN^{-k}x} - 1\right) d\mu_{S_\alpha^* f}(x);$$

and therefore

$$\begin{aligned}
& \left| \hat{\mu}_{S_\alpha^* f}(tN^{-k}) - \|S_\alpha^* f\|^2 \right| \\
\leq \quad & |t| \, N^{-k} \int_0^1 x \, d\mu_{S_\alpha^* f}(x) \\
\leq \quad & |t| \, N^{-k} \int_0^1 d\mu_{S_\alpha^* f}(x) \\
= \quad & |t| \, N^{-k} \, \|S_\alpha^* f\|^2 .
\end{aligned}$$

It follows that the difference on the left-hand side in (2.15) is estimated above

in absolute value by

$$
\begin{aligned}
& \sum_{\alpha_1,\cdots,\alpha_k} \left| e^{it\left(\frac{\alpha_1}{N}+\cdots+\frac{\alpha_k}{N^k}\right)} \right| |t| \, N^{-k} \left\| S_\alpha^* f \right\|^2 \\
= \; & |t| \, N^{-k} \sum_{\alpha_1,\cdots,\alpha_k} \left\| S_\alpha^* f \right\|^2 \\
= \; & |t| \, N^{-k} \left\langle f \mid \sum_{\alpha_1,\cdots,\alpha_k} S_\alpha S_\alpha^* f \right\rangle \\
= \; & |t| \, N^{-k} \langle f \mid f \rangle \\
= \; & |t| \, N^{-k} \|f\|^2 \\
= \; & |t| \, N^{-k}.
\end{aligned}
$$

■

Definition 2.6 *Let $k \in \mathbb{N}$, and set*

$$x_k(\alpha) := \frac{\alpha_1}{N} + \cdots + \frac{\alpha_k}{N^k} \quad \text{for } \alpha_i \in \{0, 1, \cdots, N-1\}. \tag{2.16}$$

Let (S_i) and (σ_i) be as in Corollary 2.4, and let $f \in \mathcal{H}$, $\|f\| = 1$. We set

$$\mu_f^{(k)} = \sum_{\alpha_1,\cdots,\alpha_k} \left\| S_\alpha^* f \right\|^2 \delta_{x_k(\alpha)}. \tag{2.17}$$

These measures form the sequence of measures which we use in the Riemann sum approximation of Corollary 2.5; and we are still viewing the measures μ_f and $\mu_f^{(k)}$ as measures on the unit-interval $[0, 1)$.

Corollary 2.7 *Let $N \in \mathbb{N}$, $N \geq 2$. Let (S_i) and (σ_j) be as in corollary 2.4, i.e., (S_i) is in $\operatorname{Rep}(\mathcal{O}_N, \mathcal{H})$ for some Hilbert space $\mathcal{H}$, and $\sigma_j(x) = \frac{x+j}{N}$ for $x \in [0, 1)$ and $j \in \{0, 1, \cdots, N-1\}$. Let ψ be a continuous function on $[0, 1)$, and let $k \in \mathbb{N}$. Then*

$$\left| \int_0^1 \psi \, d\mu_f - \int_0^1 \psi \, d\mu_f^{(k)} \right| \leq N^{-k} \int_{\mathbb{R}} \left| t \widehat{\psi}(t) \right| \, dt \tag{2.18}$$

where

$$\widehat{\psi}(t) = \int_0^1 \psi(x) e^{-itx} \, dx \tag{2.19}$$

is the usual Fourier transform; and we are assuming further that

$$\int_{\mathbb{R}} \left| t \widehat{\psi}(t) \right| \, dt < \infty.$$

Proof. By the Fourier inversion formula, $\psi(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{\psi}(t) e^{itx}\, dt$; and we get the following formula by a change of variables, and by the use of Fubini's theorem:

$$\int \psi \, d\mu_f - \int \psi \, d\mu_f^{(k)} = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{\psi}(t) \left(\widehat{\mu}_f(t) - \widehat{\mu_f^{(k)}}(t) \right) \, dt. \tag{2.20}$$

Since

$$\widehat{\mu_f^{(k)}}(t) = \sum_{\alpha_1, \cdots, \alpha_k} e^{itx_k(\alpha)} \left\| S_\alpha^* f \right\|^2, \tag{2.21}$$

the estimate (2.18) from Corollary 2.4 applies. An estimation of the differences in (2.20) now yields:

$$\left| \int \psi \, d\mu_f - \int \psi \, d\mu_f^{(k)} \right| \leq \frac{1}{2\pi} \int_{\mathbb{R}} \left| \widehat{\psi}(t) \right| |t| \cdot N^{-k} \, dt$$

which is the desired result. ■

Remark 2.8 *In general, a sequence of probability measures on a compact Hausdorff space X, (μ_k) is said to converge weakly to the limit μ if*

$$\lim_{k \to \infty} \int_X \psi \, d\mu_k = \int_X \psi \, d\mu \text{ for all } \psi \in C(X). \tag{2.22}$$

However, the conclusion of Corollary 2.9 for the convergence $\lim_{k\to\infty} \mu_f^{(k)} = \mu_f$ is in fact stronger than weak convergence as we will show. The notion of weak convergence of measures is significant in probability theory, see e.g., [1].

Since the measures μ_f and $\mu_f^{(k)}$ are defined on $\mathcal{B}([0,1))$, the corresponding distribution functions F_f and $F_f^{(k)}$ are defined for $x \in [0,1)$ as follows

$$F_f(x) = \mu_f([0,x]) \text{ and } F_f^{(k)}(x) = \mu_f^{(k)}([0,x]). \tag{2.23}$$

Corollary 2.9 *Let (S_i) and (σ_i) be as in Corollary 2.7. Let $f \in \mathcal{H}$, $\|f\| = 1$, be given, and let μ_f resp., $\mu_f^{(k)}$ be the corresponding measures, with distribution functions F_f and $F_f^{(k)}$, respectively. Then*

$$\lim_{k \to \infty} F_f^{(k)}(x) = F_f(x). \tag{2.24}$$

Proof. We already proved that the sequence of measures $\mu_f^{(k)}$ converges weakly to μ_f as $k \to \infty$. Furthermore, it is known that weak convergence $\mu_f^{(k)} \to \mu_f$ implies that (2.24) holds whenever x is a point of continuity for F_f, see [1, Theorem 2.3, p.5]. (For the case of the wavelet representations, it is known that every x is a point of continuity, but Example 1.3 shows that the measures μ_f are not continuous in general.) The argument from the proof of Corollary 2.4 shows

that, in general, the points of discontinuity of $F_f(\cdot)$ must lie in the set (2.16) of N-adic fractions. Using Theorem 2.2 and formula (2.17), we conclude that if $x_k(\alpha)$ is a point of discontinuity of $F_f(\cdot)$, then $\left|F_f^{(k+n)}(x_k(\alpha)) - F_f(x_k(\alpha))\right| \leq N^{-k-n}$, and therefore

$$\lim_{n\to\infty} F_f^{(k+n)}(x_k(\alpha)) = F_f(x_k(\alpha))$$

which is the desired conclusion, see (2.24). ■

3 The Measures μ_f

In the previous section, we showed that the decomposition theory for representations of the Cuntz algebra $\mathcal{O}_N$ may be analyzed by the use of projection valued measures on a class of iterated function systems (IFS). It is known that there is a simple C^*-algebra $\mathcal{O}_N$ for each $N \in \mathbb{N}$, $N \geq 2$, such that the representations of $\mathcal{O}_N$ are in on-one correspondence with systems of isometries (S_i) which satisfy the two relations (1.7)–(1.8). The C^*-algebra $\mathcal{O}_N$ *is* defined abstractly on N generators s_i which satisfy

$$\sum_{i=0}^{N-1} s_i s_i^* = 1 \text{ and } s_i^* s_j = \delta_{i,j}1. \tag{3.1}$$

A representation ρ of $\mathcal{O}_N$ on a Hilbert space $\mathcal{H}$ is a $*$-homomorphism from $\mathcal{O}_N$ into $B(\mathcal{H})$ = the algebra of all bounded operators on $\mathcal{H}$. The set of representations acting on $\mathcal{H}$ is denoted $\mathrm{Rep}(\mathcal{O}_N, \mathcal{H})$. The connection between ρ and the corresponding (S_i)-system is fixed by $\rho(s_i) = S_i$. While the subalgebra $\mathcal{C}$ in $\mathcal{O}_N$ generated by the monomials $s_{i_j} \cdots s_{i_k} s_{i_k}^* \cdots s_{i_1}^*$ is maximal abelian in $\mathcal{O}_N$, the von Neumann algebra $\mathfrak{A}$ generated by $\rho(\mathcal{C})$ may not be maximally abelian in $B(\mathcal{H})$. Whether it is, or not, depends on the representation. It is known to be maximally abelian if the operators $S_i = \rho(s_i)$ are given by (1.6), and if the functions m_i satisfy the usual subband conditions from wavelet theory. For details, see [5] and [2]. For these representations, $\mathcal{H} = L^2(\mathbb{T})$; and the representations define wavelets

$$\psi_{i,j,k}(x) \colon = N^{\frac{j}{2}} \psi_i(N^j x - k),\ i = 1, \cdots, N-1,\ \ j,k \in \mathbb{Z} \tag{3.2}$$

in $L^2(\mathbb{R})$. Such wavelets are specified by the functions $\psi_1, \cdots, \psi_{N-1} \in L^2(\mathbb{R})$. These representations ρ are called *wavelet representations.* The assertion is that, if $\mathfrak{A}$ is defined by a wavelet representation, then $\mathfrak{A}$ is maximally abelian in $B(L^2(\mathbb{T}))$. The operators commuting with $\mathfrak{A}$ are denoted $\mathfrak{A}'$, and it is easy to see that $\mathfrak{A}$ is maximally abelian if and only if $\mathfrak{A}'$ is abelian. An abelian von Neumann algebra $\mathfrak{A} \subset B(\mathcal{H})$ is said to have a cyclic vector f if the closure of $\mathfrak{A}f$ is $\mathcal{H}$. For $f \in \mathcal{H}$, the closure of $\mathfrak{A}f$ is denoted $\mathcal{H}_f$. It is known that $\mathfrak{A}$ has a cyclic vector if and only if it is maximally abelian. Clearly, if f is a cyclic vector, then the measure $\mu_f(\cdot) \colon = \|P(\cdot)f\|^2$ determines the other measures $\{\mu_g \mid g \in \mathcal{H}\}$.

Lemma 3.1 *Let $\mathfrak{A} \subset B(\mathcal{H})$ be an abelian C^*-algebra, and let ρ: $C(X) \cong \mathfrak{A}$ be the Gelfand representation, X a compact Hausdorff space. Let $f \in \mathcal{H}$, $\|f\| = 1$.*
(a) Then there is a unique Borel measure μ on X, and an isometry V_f: $L^2(\mu) \to \mathcal{H}$, such that

$$V_f(1) = f, \tag{3.3}$$

$$V_f(\psi) = \rho(\psi)f \quad , \psi \in C(X), \tag{3.4}$$

and

$$V_f(L^2(\mu)) = \mathcal{H}_f. \tag{3.5}$$

(b) Let $f_i \in \mathcal{H}$, $\|f_i\| = 1$, $i = 1, 2$, and suppose $\mu_1 << \mu_2$. Setting $k = \frac{d\mu_1}{d\mu_2}$ where $\mu_i\colon = \mu_{f_i}$, $i = 1, 2$, then $U\psi = \sqrt{k}\psi$ defines an isometry U: $L^2(\mu_1) \to L^2(\mu_2)$, and $W\colon = V_{f_2}UV_{f_1}^*\colon \mathcal{H}_{f_1} \to \mathcal{H}_{f_2}$ is in the commutant of $\mathfrak{A}$.

Proof. Part (a) follows from the spectral theorem applied to abelian C^*-algebras, see e.g., [8]. To prove (b), let f_i be the two vectors in $\mathcal{H}$, and set $\mu_i\colon = \mu_{f_i}$, i.e., the corresponding measures on X. Since $\mu_1 << \mu_2$, the Radon-Nikodym derivative $k\colon = \frac{d\mu_1}{d\mu_2}$ is well defined. Clearly then

$$\|U\psi\|^2_{L^2(\mu_2)} = \int_X |\psi|^2 \, k \, d\mu_2 = \int_X |\psi|^2 \, d\mu_1 = \|\psi\|_{L^2(\mu_1)} .$$

As a result $W\colon = V_{f_2}UV_{f_1}^*$ is a well defined partial isometry in $\mathcal{H}$. For $\psi \in C(X)$, we compute

$$\begin{aligned} W\rho(\psi) &= V_{f_2}UV_{f_1}^*\rho(\psi) \\ &= V_{f_2}UM_\psi V_{f_1}^* \\ &= V_{f_2}M_\psi UV_{f_1}^* \\ &= \rho(\psi)V_{f_2}UV_{f_1}^* \\ &= \rho(\psi)W, \end{aligned}$$

and we conclude that $W \in \mathfrak{A}'$. The prime stands for commutant. ∎

Theorem 3.2 *Let $N \in \mathbb{N}$, $N \geq 2$; let $\mathcal{H}$ be a Hilbert space, and (S_i) a representation of $\mathcal{O}_N$ in $\mathcal{H}$. Let (X, d) be a compact metric space, and $(\sigma_i)_{0 \leq i < N}$ an iterated function system which is complete and non-overlapping. Let P: $\mathcal{B}(X) \to B(\mathcal{H})$ the corresponding projection valued measure. Suppose the von Neumann algebra $\mathfrak{A}$ generated by $\{S_\alpha S_\alpha^* \mid k \in \mathbb{N},\ \alpha \in (\mathbb{Z}_N)^k\}$ is maximally abelian. For $f \in \mathcal{H}$, $\|f\| = 1$, set $\mu_f(\cdot)\colon = \|P(\cdot)f\|^2$. Then the following two conditions are equivalent:*
(i) f is a cyclic vector for $\mathfrak{A}$.
(ii) $\mu_f \circ \sigma_i^{-1} << \mu_f$, $i = 0, 1, \cdots, N-1$.

Proof. We first claim that

$$\mu_f \circ \sigma_i^{-1} = \mu_{S_i f}. \tag{3.6}$$

To see this, we apply (2.4) to $S_i f$. Then $\mu_{S_i} f = \sum_j \mu_{S_j^* S_i f} \circ \sigma_j^{-1} = \mu_f \circ \sigma_i^{-1}$ since $S_j^* S_i f = \delta_{i,j} f$. This is the desired identity (3.6).

Secondly, let $i \neq j$. then naturally $S_i f \perp S_j f$. But we claim that

$$\mathcal{H}_{S_i f} \perp \mathcal{H}_{S_j f}; \tag{3.7}$$

i.e., for all $A \in \mathfrak{A}$, $\langle S_i f \mid A S_j f \rangle = 0$. Since $\mathfrak{A}$ is generated by the projections $S_\alpha S_\alpha^*$, it is enough to show that $S_i^* S_\alpha S_\alpha^* S_j = 0$ for $\alpha = (\alpha_1, \cdots, \alpha_k)$. But $S_i^* S_\alpha S_\alpha^* S_j = \delta_{i,\alpha_1} \delta_{j,\alpha_1} S_{\alpha_2} \cdots S_{\alpha_k} S_{\alpha_k}^* \cdots S_{\alpha_2}^* = 0$ since $i = j$. The orthogonality relation (3.7) follows.

We first prove (i)$\Longrightarrow$(ii); in fact we prove that $\mu_g << \mu_f$ for all $g \in \mathcal{H}$, if f is assumed cyclic. If f is cyclic, and $g \in \mathcal{H}$, $\|g\| = 1$, then clearly $\mathcal{H}_g \subset \mathcal{H}_f$. By the argument in Lemma 3.1(b), we conclude that $W\colon = V_f^* V_g\colon L^2(\mu_g) \to L^2(\mu_f)$ commutes with the multiplication operators. Setting $k\colon = W(1)$, we have $\int_X |\psi|^2 \, d\mu_g = \int_X |W\psi|^2 \, d\mu_f = \int |\psi|^2 |k|^2 \, d\mu_f$, or equivalently, $d\mu_g = |k|^2 d\mu_f$. The conclusion $d\mu_g << d\mu_f$ follows, and $\frac{d\mu_g}{d\mu_f} = |k|^2$.

To prove (ii)$\Longrightarrow$(i); let $i, j \in \mathbb{Z}_N$, and suppose $i \neq j$. We saw that then $\mathcal{H}_{S_i f} \perp \mathcal{H}_{S_j f}$. Suppose f is not cyclic. Since by (3.6) $\mu_{S_i f} = \mu_f \circ \sigma_i^{-1}$, we get the two isometries $V_{S_i f}\colon L^2(\mu_f \circ \sigma_i) \to \mathcal{H}_{S_i f}$ with orthogonal ranges. Let

$$k_i = \frac{d\mu_f \circ \sigma_i^{-1}}{d\mu_f}, \text{ and } k_j = \frac{d\mu_f \circ \sigma_j^{-1}}{d\mu_f}.$$

Set

$$U_i \psi = \psi \sqrt{k_i} \text{ and } U_j \psi = \psi \sqrt{k_j}.$$

Then the following operator

$$W\colon = V_{S_j f} U_j^* U_i V_{S_i f}^* \tag{3.8}$$

is well defined. It is a partial isometry in $\mathcal{H}$ with initial space $\mathcal{H}_{S_i f}$ and final space $\mathcal{H}_{S_j f}$; i.e., $W^* W = \operatorname{proj}(\mathcal{H}_{S_i f}) = p_i$, and $W W^* = \operatorname{proj}(\mathcal{H}_{S_j f}) = p_j$. By the lemma, W is in the commutant of $\mathfrak{A}$. But the two projections p_i and p_j are orthogonal by the lemma, i.e., $p_i p_j = 0$. Relative to the decomposition $p_i \mathcal{H} \oplus p_j \mathcal{H}$, we now consider the following two block matrix operators

$$\begin{pmatrix} 0 & 0 \\ W & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & W^* \\ 0 & 0 \end{pmatrix};$$

and note that

$$\begin{pmatrix} 0 & 0 \\ W & 0 \end{pmatrix} \begin{pmatrix} 0 & W^* \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & p_j \end{pmatrix},$$

while

$$\begin{pmatrix} 0 & W^* \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ W & 0 \end{pmatrix} = \begin{pmatrix} p_i & 0 \\ 0 & 0 \end{pmatrix}.$$

Since the two non-commuting operators are in $\mathfrak{A}'$, it follows that $\mathfrak{A}'$ is non-abelian, and as a result that $\mathfrak{A}$ is not maximally abelian. ■

Two Examples: (a) Let $\mathcal{H} = L^2(\mathbb{T})$ where as usual $\mathbb{T}$ denotes the torus, equipped with Haar measure. Set $e_n(z) := z^n$, $z \in \mathbb{T}$, $n \in \mathbb{Z}$, and define

$$\begin{cases} S_0 f(z) = f(z^2) \text{ for } f \in \mathcal{H}, \text{ and } z \in \mathbb{T} \\ S_1 f(z) = z f(z^2). \end{cases} \tag{3.9}$$

As noted in Section 1, this system is in $\mathrm{Rep}(\mathcal{O}_2, \mathcal{H})$. By Theorem 2.2, there is a unique projection valued measure $P(\cdot)$ on $\mathcal{B}([0,1))$ such that

$$P\left(\left[\frac{\alpha_1}{2} + \cdots + \frac{\alpha_k}{2^k}, \frac{\alpha_1}{2} + \cdots + \frac{\alpha_k}{2^k} + \frac{1}{2^k}\right)\right) = S_\alpha S_\alpha^* \tag{3.10}$$

where $S_\alpha = S_{\alpha_1} \cdots S_{\alpha_k}$.

It is easy to check that the range of the projection $S_\alpha S_\alpha^*$ is the closed subspace in $\mathcal{H}$ spanned by

$$\left\{ e_n \mid n = \alpha_1 + 2\alpha_2 + \cdots + 2^{k-1}\alpha_k + 2^k p, \, p \in \mathbb{Z} \right\},$$

and it follows that

$$S_\alpha S_\alpha^* e_0 = \begin{cases} e_0 \text{ if } \alpha_1 = \alpha_2 = \cdots = \alpha_k = 0 \\ 0 \text{ otherwise.} \end{cases}$$

Hence

$$\mathcal{H}_{e_0} = [\mathfrak{A}_{e_0}] = \mathbb{C} e_0$$

is one-dimensional, and

$$\mu_{e_0}(\cdot) = \|P(\cdot) e_0\|^2 = \delta_0$$

where δ_0 is the Dirac measure on $[0,1)$ at $x = 0$. With the IFS $\sigma_0(x) = \frac{x}{2}$, $\sigma_1(x) = \frac{x+1}{2}$ on the unit-interval, we get

$$\left[\begin{array}{l} \mu_{e_0} \circ \sigma_0^{-1} = \delta_0 \\ \mu_{e_0} \circ \sigma_1^{-1} = \delta_{\frac{1}{2}} \end{array} \right. \tag{3.11}$$

making it clear that condition (ii) in Theorem 3.2 is not satisfied.

(b) We now modify (3.9) as follows:

Set

$$\begin{cases} S_0 f(z) = \frac{1}{\sqrt{2}}(1+z) f(z^2) \text{ for } f \in L^2(\mathbb{T}), \text{ and } z \in \mathbb{T} \\ S_1 f(z) = \frac{1}{\sqrt{2}}(1-z) f(z^2). \end{cases} \tag{3.12}$$

This system (S_i) is in $\mathrm{Rep}(\mathcal{O}_2, L^2(\mathbb{T}))$, and $\mu_{e_0}(\cdot) = \|P(\cdot)e_0\|^2$ = Lebesgue measure dt on $[0,1)$, where $P(\cdot)$ is again determined by (3.10). This is the representation of O_2 which corresponds to the usual Haar wavelet, i.e., to

$$\psi(x) = \begin{cases} 1 \text{ if } 0 \leq x < \frac{1}{2} \\ -1 \text{ if } \frac{1}{2} \leq x < 1 \end{cases} \tag{3.13}$$

and

$$\psi_{j,k}(x) = 2^{\frac{j}{2}}\psi(2^j x - k) \text{ for } j,k \in \mathbb{Z} \tag{3.14}$$

is then the standard Haar basis for $L^2(\mathbb{R})$; compare this with (3.2). For this representation $\mathfrak{A}$ can be checked to be maximally abelian, but it also follows from the theorem, since now the analog of (3.11) is

$$\begin{cases} \mu_{e_0} \circ \sigma_0^{-1} = 2dt \text{ restricted to } [0, \frac{1}{2}) \\ \mu_{e_0} \circ \sigma_1^{-1} = 2dt \text{ restricted to } [\frac{1}{2}, 1). \end{cases}$$

Since $\mu_{e_0} = dt$ restricted to $[0,1)$, it is clear that now condition (ii) in Theorem 3.2 is satisfied.

References

[1] Patrick Billingsley, Weak convergence of measures: Applications in Probability. CBMS, vol. 5, SIAM Philadelphia, 1971.

[2] O. Bratteli, P. Jorgensen, "Wavelets Through A Looking Glass", Birkhäuser, 2002.

[3] J. Cuntz, Simple C^*-algebras generated by isometries, *Comm. Math. Phys.* **57** (1977), 173–185.

[4] J. E. Hutchinson, Fractals and self-similarity, *Indiana Univ. Math. J.* **30** (1981), 713–747.

[5] Palle E. T. Jorgensen, Minimality of the data in wavelet filters, *Advances in Math* **159** (2001) 143–228.

[6] P. E. T. Jorgensen, Measures in wavelet decompositions, preprint, University of Iowa, 2003.

[7] P. E. T. Jorgensen, Matrix factorizations, algorithms, wavelets, *Notices of the Amer. Math. Soc.* **50** (2003) 880–984.

[8] Edward Nelson, "Topics in Dynamics I: Flows", Princeton University Press, 1970.